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On the space of quadruples of quinary alternating forms

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Abstract

We discuss some aspects of the invariant theory and arithmetic of the prehomogeneous vector space of quadruples of quinary alternating forms. In particular, we complete the explicit construction of all prehomogeneous covariants of this space and give the rational classification of maximal singular orbits.

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1. Introduction

The space V of quadruples of quinary alternating forms under the action of $G = \mathrm{GL}(4) \times \mathrm{GL}(5)$ is one of the largest and most complicated of the exceptional examples to be found on the Sato–Kimura list [9] of irreducible, regular, reduced prehomogeneous vector spaces. It has attracted attention from various points of view. The orbit decomposition over \mathbb{C} , microlocal structure and b -function have been determined by Ozeki [7,8]. In [12], it was shown that the set of non-singular orbits

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in V over a field k is in one-to-one correspondence with the set of isomorphism classes of separable quintic k -algebras, thus revealing the arithmetic significance of the space. Recently, the authors have discovered a refinement of this correspondence [5] and used it [6] to obtain an upper bound on the number of quintic number fields with bounded discriminant. Our purpose here is to continue the investigation of the invariant theory and arithmetic of V and the interaction between them.

In [3], the first author determined a complete list of the relatively equivariant maps from (G, V) to any other prehomogeneous vector space and showed how all these maps could be derived from two maps, one into the space of quaternary quadratic forms and the other into the space of quinary quadratic forms. The first of these was originally constructed explicitly by Amano et al. [1]. A second construction was given in [5] and will be reviewed in Section 4. In the same section, we give an explicit construction of the second map, thereby completing the construction of the prehomogeneous covariants of (G, V) . We also construct several other covariants that are required in later sections.

For arithmetic purposes, it is important to be able to recognize when a relatively equivariant map is defined over \mathbb{Z} . Previously, in [5,6], the authors have addressed this problem by a combination of symbolic computation using [11] and ad hoc arguments. In Section 2, we give a general criterion for recognizing integrality of relatively equivariant maps from a prehomogeneous vector space that reduces the problem to a numerical, as opposed to symbolic, computation. This result could, incidentally, be used to streamline a number of arguments in the authors' earlier work.

The correspondence between non-singular orbits in V over a field k and isomorphism classes of separable quintic k -algebras was described in [12] in two ways, via Galois cohomology and via geometry. One consequence of the results of [5,6] is a description of this correspondence in terms of invariant theory. Specifically, the structure constants of the quintic algebra corresponding to a point in V are explicitly given in terms of certain covariants of (G, V) defined over \mathbb{Z} . This construction also extends the map by associating a non-separable quintic algebra to each singular orbit. In Section 5, we review this construction and then give a new invariant-theoretic construction that associates a sextic algebra to each orbit in V . If v is a point in V then we denote the associated quintic algebra by \tilde{R}_v and the associated sextic algebra by \tilde{S}_v . For non-singular points v , the isomorphism class of \tilde{R}_v determines the orbit of v and hence the isomorphism class of \tilde{S}_v . We make the map $\tilde{R}_v \mapsto \tilde{S}_v$ explicit in Theorem 1, where it is shown that when v is non-singular, \tilde{S}_v is isomorphic to the sextic resolvent algebra of \tilde{R}_v .

In Section 6, we consider the set Y' of smooth points on the singular hypersurface $Y \subset V$. We refer to Y' as the *subgeneric orbit*; this terminology is justified in Proposition 3, which states that Y'_K is a single G_K -orbit whenever K is an algebraically closed field of characteristic zero. We prove this by a conceptual argument that may be adapted to other prehomogeneous vector spaces and avoids the need for extensive calculation. We then consider the classification of the orbit space $G_k \backslash Y'_k$ when k is not algebraically closed. The result is stated in Theorem 2; it emerges that $G_k \backslash Y'_k$ is in one-to-one correspondence with the set of isomorphism classes of

separable cubic k -algebras and that this correspondence may be realized using the algebra \tilde{S}_v .

2. An integrality criterion

In this section, we shall prove a useful integrality criterion for covariants of prehomogeneous vector spaces defined over \mathbb{Z} . Since the denotation of the term “prehomogeneous vector space” varies somewhat in the literature, we shall first describe precisely which spaces we wish to consider.

Let G be an affine algebraic group, V an affine space and $\rho: G \rightarrow \mathrm{GL}(V)$ a morphism via which G acts on V . Assume that these three objects are defined over \mathbb{Z} . In all subsequent notation, the morphism ρ will be suppressed. We call the pair (G, V) a *prehomogeneous vector space defined over \mathbb{Z}* if, for every algebraically closed field k , the space V_k contains a Zariski open G_k -orbit. If such an orbit exists then it is unique and we say that a point $v \in V_k$ is *generic* if it lies in this orbit. We say that $w \in V_{\mathbb{Z}}$ is *universally generic* if the image of w in V_k is generic for every algebraically closed field k .

If R is a commutative ring with 1 then we may similarly define the notion of a prehomogeneous vector space (G, V) defined over R . We simply consider the set of all pairs (v, k) , where k is an algebraically closed field and $v: R \rightarrow k$ is a homomorphism, regard k as an R -algebra via v , and require that $(k \times_R V)_k$ contain a Zariski open $(k \times_R G)_k$ -orbit.

Proposition 1. *Let (G, V) be a prehomogeneous vector space defined over \mathbb{Z} and $w \in V_{\mathbb{Z}}$ a universally generic point. Suppose that (π, W) is a rational representation of G defined over \mathbb{Z} . Suppose that $\Phi: \mathbb{Q} \times_{\mathbb{Z}} V \rightarrow \mathbb{Q} \times_{\mathbb{Z}} W$ is a $\mathbb{Q} \times_{\mathbb{Z}} G$ -equivariant morphism and that $\Phi(w) \in W_{\mathbb{Z}}$. Then Φ is (derived from a morphism) defined over \mathbb{Z} .*

Proof. We may choose an ordered \mathbb{Z} -basis ξ_1, \dots, ξ_m for $W_{\mathbb{Z}}$ and express Φ uniquely as

$$\Phi(x) = \sum_{i=1}^m \Phi_i(x) \xi_i,$$

where $\Phi_i(x) \in \mathbb{Q}[V]$. It is sufficient to show that $\Phi_i(x) \in \mathbb{Z}[V]$ for all i . Let N be the least positive integer such that $N\Phi_i(x) \in \mathbb{Z}[V]$ for all i ; we wish to show that $N = 1$. Suppose to the contrary and choose some prime p that divides N . Let \mathcal{O} be the ring of all algebraic integers, \mathfrak{P} a prime ideal of \mathcal{O} above p and $\mathbb{F} = \mathcal{O}/\mathfrak{P}$. Note that \mathbb{F} is an algebraically closed field. We may regard $\Psi = N\Phi$ as a G -equivariant morphism $\Psi: V \rightarrow W$. It induces a $G_{\mathbb{F}}$ -equivariant map $\bar{\Psi}: V_{\mathbb{F}} \rightarrow W_{\mathbb{F}}$. Since $\Phi(w) \in W_{\mathbb{Z}}$ and $N \in \mathfrak{P}$, $\bar{\Psi}(w) = 0$, and so $\bar{\Psi}(v) = 0$ for all v in the $G_{\mathbb{F}}$ -orbit of w . By hypothesis, $w \in V_{\mathbb{F}}$ is generic and it follows that $\bar{\Psi} = 0$. Now the image of the list ξ_1, \dots, ξ_m in $W_{\mathbb{F}}$ is an ordered basis for $W_{\mathbb{F}}$ and we conclude that the coefficients of $N\Phi_i(x)$ must lie in \mathfrak{P} for all i . But the coefficients are also in \mathbb{Z} and so they are in $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$. Thus

$(N/p)\Phi_i(x) \in \mathbb{Z}[V]$ for all i and hence N was not minimal. This contradiction shows that $N = 1$. \square

3. Quadruples of quinary alternating forms

The purpose of this section is to establish our notation connected with the space of quadruples of quinary alternating forms and to review some known properties of the space. Let $\text{Aff}(n)$ denote n -dimensional affine space and $\text{GL}(n)$ denote the $n \times n$ general linear group, both regarded as varieties over \mathbb{Z} . Then $\text{GL}(n)$ acts on $\text{Aff}(n)$ and the action is defined over \mathbb{Z} . Let $V = \text{Aff}(4) \otimes \wedge^2 \text{Aff}(5)$ and $G = \text{GL}(4) \times \text{GL}(5)$ with the induced \mathbb{Z} -structures and the natural action of G on V . We denote the standard ordered basis of $\text{Aff}(4)$ by f_1, \dots, f_4 and that of $\text{Aff}(5)$ by e_1, \dots, e_5 . We shall systematically use roman letters for indices ranging from 1 to 4 and greek letters for indices ranging from 1 to 5. We introduce canonical coordinates $x_{i\alpha\beta}$ on V dual to the basis

$$\{f_i \otimes (e_\alpha \wedge e_\beta) \mid 1 \leq i \leq 4, 1 \leq \alpha < \beta \leq 5\}$$

and extend the notation by setting $x_{i\alpha\alpha} = 0$ and $x_{i\beta\alpha} = -x_{i\alpha\beta}$. For $h \in \text{GL}(4)$ and $g \in \text{GL}(5)$, we write $h = (h_i^j)$ and $g = (g_\alpha^\beta)$, where the subscript is the row index and the superscript the column index. There is an induced action of G on the coordinate ring of V , given explicitly on the extended canonical coordinates by

$$(g, h)x_{i\alpha\beta} = \sum x_{j\gamma\delta} h_i^j g_\alpha^\gamma g_\beta^\delta. \quad (1)$$

In this formula, we use a variation of the summation convention: when a sum is specified without explicit summation indices, the sum is taken over all indices that appear in pairs, once as a superscript and once as a subscript. Thus, in (1), the sum is over j, γ and δ . We shall continue to employ this convention throughout.

Let

$$w_1 = f_1 \otimes (e_1 \wedge e_2),$$

$$w_2 = f_2 \otimes (e_3 \wedge e_4),$$

$$w_3 = f_3 \otimes (e_1 \wedge e_5 + e_3 \wedge e_5),$$

$$w_4 = f_4 \otimes (-e_1 \wedge e_2 + e_1 \wedge e_4 + e_2 \wedge e_4 + e_2 \wedge e_5 - e_4 \wedge e_5)$$

and $w = w_1 + w_2 + w_3 + w_4$. It is proved in [12] that (G, V) is a prehomogeneous vector space defined over \mathbb{Z} and that $w \in V_{\mathbb{Z}}$ is a universally generic point.

It is known that there is an irreducible relatively invariant polynomial $P \in \mathbb{Q}[V]$ of degree 40. In both [1] and [9], the construction of P is carried out over \mathbb{C} , but it is clear from the expressions given in these references that P may be taken to have coefficients in \mathbb{Q} . This will also follow from the results of Section 4. Let d_1 and d_2 be the characters of G defined by $d_1(h, g) = \det(h)$ and $d_2(h, g) = \det(g)$ and set $\omega = d_1^5 d_2^8$. Then P transforms under G by $P(gx) = \omega^2(g)P(x)$. The rational representation of G associated to ω^2 is defined over \mathbb{Z} and it follows from Proposition 1 that if we normalize P so that $P(w) = 1$ then $P \in \mathbb{Z}[V]$. We use this normalization from

now on. Let Y be the hypersurface defined by the equation $P = 0$ and V^{ss} be the complement of Y .

4. Construction of covariants

By a *covariant* (or *concomitant*) of (G, V) over a commutative ring R with 1, we mean an $R \times_{\mathbb{Z}} G$ -equivariant morphism $\Phi: R \times_{\mathbb{Z}} V \rightarrow W$, where (π, W) is a rational representation of $R \times_{\mathbb{Z}} G$ defined over R . In this section we shall construct several covariants of (G, V) over \mathbb{Z} and discuss their properties and significance. The constructions will be made using the language of tensor invariant theory. We now review what little we need from this theory.

If $g \in \text{GL}(n)$ then we write $g^{-1} = (\bar{g}_i^j)$. We let ε denote the fully alternating tensor of rank n with components

$$\varepsilon^{i_1 \dots i_n} = \begin{cases} 1 & \text{if } (i_1, \dots, i_n) \text{ is an even rearrangement of } (1, \dots, n), \\ -1 & \text{if } (i_1, \dots, i_n) \text{ is an odd rearrangement of } (1, \dots, n), \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, we have

$$\sum \varepsilon^{i_1 \dots i_n} g_{i_1}^{j_1} \dots g_{i_n}^{j_n} = \det(g) \varepsilon^{j_1 \dots j_n}, \quad (2)$$

$$\det(g) \sum \varepsilon^{j_1 \dots j_{n-1} c} \bar{g}_c^b = \sum \varepsilon^{i_1 \dots i_{n-1} b} g_{i_1}^{j_1} \dots g_{i_{n-1}}^{j_{n-1}} \quad (3)$$

for $g \in \text{GL}(n)$, where all indices range from 1 to n . Of these, (2) is simply the usual definition of the determinant polynomial and (3) expresses the relationship between the inverse and the classical adjoint of g .

Consider the polynomials on V defined as follows:

$$C_{jk}^l(x) = \frac{1}{2^5} \sum \varepsilon^{\alpha_1 \beta_1 \alpha_4 \beta_4 \alpha_3} \varepsilon^{\beta_3 \alpha_2 \beta_2 \alpha_5 \beta_5} \varepsilon^{i_3 i_4 i_5 l} x_{j \alpha_1 \beta_1} x_{k \alpha_2 \beta_2} x_{i_3 \alpha_3 \beta_3} x_{i_4 \alpha_4 \beta_4} x_{i_5 \alpha_5 \beta_5},$$

$$Q_{\zeta \eta \kappa}^{\rho}(x) = \frac{1}{2} \sum \varepsilon^{i_1 i_2 i_3 i_4} \varepsilon^{\rho \beta_1 \beta_2 \alpha_4 \beta_4} x_{i_1 \zeta \beta_1} x_{i_2 \eta \beta_2} x_{i_3 \iota \kappa} x_{i_4 \alpha_4 \beta_4},$$

$$Z_{\mu \lambda}^v(x) = \frac{1}{2^2} \sum \varepsilon^{\beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2} Q_{\mu \alpha_1 \beta_1 \gamma_1}^v(x) Q_{\lambda \alpha_2 \beta_2 \gamma_2}^{\alpha_1}(x),$$

$$\Gamma_v^{\mu \lambda}(x) = \frac{1}{2^5 3} \sum \varepsilon^{\beta_1 \gamma_1 \delta_1 \alpha_2 \delta_2} \varepsilon^{\beta_2 \gamma_2 \beta_3 \gamma_3 \delta_3} Q_{\alpha_1 \beta_1 \gamma_1 \delta_1}^{\mu}(x) Q_{\alpha_2 \beta_2 \gamma_2 \delta_2}^{\lambda}(x) Q_{\nu \beta_3 \gamma_3 \delta_3}^{\alpha_1}(x),$$

$$D_{ij}(x) = \frac{1}{3} \sum C_{ir}^s(x) C_{sj}^r(x),$$

$$A_{\mu \lambda}(x) = \frac{1}{2^2 3} \sum Z_{\mu \alpha}^{\beta}(x) Z_{\beta \lambda}^{\alpha}(x),$$

$$\Delta^{\alpha \beta}(x) = \frac{1}{2^2} \sum \Gamma_{\delta}^{\alpha \gamma}(x) \Gamma_{\gamma}^{\delta \beta}(x).$$

The significance of these expressions may not immediately be clear. As we shall show, each of them corresponds to a covariant of the space (G, V) and certain of these covariants enjoy remarkable algebraic properties. We note that the polynomials $C_{jk}^l(x)$ and $D_{ij}(x)$ were introduced and studied in Section 5 of [5]. Next, we record the transformation laws obeyed by these families of polynomials; they say, in short, that each family defines a relative tensor of the indicated type and some weight.

Lemma 1. For $(h, g) \in G$ we have

$$\begin{aligned} C_{jk}^l((h, g)x) &= \det(h)\det(g)^2 \sum C_{ab}^c(x) h_j^a h_k^b \bar{h}_c^l, \\ Q_{\zeta\eta\kappa}^\rho((h, g)x) &= \det(h)\det(g) \sum Q_{\alpha\beta\gamma\delta}^\epsilon(x) g_\zeta^\alpha g_\eta^\beta g_\kappa^\gamma g_\epsilon^\delta \bar{g}_\rho^\epsilon, \\ Z_{\mu\lambda}^v((h, g)x) &= \det(h)^2 \det(g)^3 \sum Z_{\alpha\beta}^\gamma(x) g_\mu^\alpha g_\lambda^\beta \bar{g}_\gamma^v, \\ \Gamma_v^{\mu\lambda}((h, g)x) &= \det(h)^3 \det(g)^5 \sum \Gamma_\gamma^{\alpha\beta}(x) g_v^\gamma \bar{g}_\alpha^\mu \bar{g}_\beta^\lambda, \\ D_{ij}((h, g)x) &= \det(h)^2 \det(g)^4 \sum D_{ab}(x) h_i^a h_j^b, \\ A_{\mu\lambda}((h, g)x) &= \det(h)^4 \det(g)^6 \sum A_{\alpha\beta}(x) g_\mu^\alpha g_\lambda^\beta, \\ A^{\gamma\delta}((h, g)x) &= \det(h)^6 \det(g)^{10} \sum A^{\alpha\beta}(x) \bar{g}_\alpha^\gamma \bar{g}_\beta^\delta. \end{aligned}$$

Proof. It is routine to use (1)–(3) to establish these identities. \square

Let $d_1 d_2^2 \otimes \text{Aff}(4)^* \otimes \text{Aff}(4) \otimes \text{Aff}(4)$ denote the \mathbb{Z} -variety $\text{Aff}(4)^* \otimes \text{Aff}(4) \otimes \text{Aff}(4)$ with the standard action of G twisted by the character $d_1 d_2^2$. It follows from Lemma 1 that there is a covariant

$$C: V \rightarrow d_1 d_2^2 \otimes \text{Aff}(4)^* \otimes \text{Aff}(4) \otimes \text{Aff}(4)$$

defined by

$$C(v) = \sum_{j,k,l} C_{jk}^l(v) f_l^* \otimes f_j \otimes f_k.$$

It is clear from the definition of $C_{jk}^l(x)$ that this covariant is defined over \mathbb{Q} . In a similar way, there are covariants Q , Z , Γ , D , A and Δ associated to the other families of polynomials.

Lemma 2. The covariants C , Q , Z , Γ , D , A and $\frac{1}{2}\Delta$ are defined over \mathbb{Z} and non-zero.

Proof. By Proposition 1, all we need do is verify that each of these covariants takes a non-zero integral value at w . This may be done by direct calculation using, for example, [11]. \square

The reason for normalizing Δ so that it is not primitive will become apparent in Section 5.

In addition to the transformation laws recorded in Lemma 1, the various covariants also have symmetry properties. In particular, it is clear from the definitions that D , Δ and Δ take their values in $\text{sym}^2(\text{Aff}(4))$, $\text{sym}^2(\text{Aff}(5))$ and $\text{sym}^2(\text{Aff}(5)^*)$, respectively. In Section 5 of [3], all prehomogeneous covariants of (G, V) were classified and it was shown how all these covariants could be derived from two fundamental prehomogeneous covariants. These two fundamental covariants are the unique covariant of degree 10 mapping to $d_1^2 d_2^4 \otimes \text{sym}^2(\text{Aff}(4))$ and the unique covariant of degree 16 mapping to $d_1^4 d_2^6 \otimes \text{sym}^2(\text{Aff}(5))$. In our current notation, these covariants are D and Δ , respectively. In particular, the claim concerning the value of $\Delta(w)$ made in [3] may be verified by direct calculation using the expression for Δ given above.

The covariants constructed here are connected to each other and to P by numerous identities. Indeed, Amano, Fujigami and Kogiso originally constructed D , by a different method, in order to derive a convenient expression for P . We next record several of these identities; the symbol δ is the Kronecker delta.

Lemma 3. *We have*

- (1) $\det(D_{ij}(x)) = 5^3 P(x)$.
- (2) $\det(A_{\mu\lambda}(x)) = 2^4 3 P(x)^2$.
- (3) $\det(\Delta^{\alpha\beta}(x)) = 2^6 3^4 P(x)^3$.
- (4) $\sum \Delta_{\alpha\beta}(x) \Delta^{\beta\gamma}(x) = 2^2 3 P(x) \delta_{\alpha}^{\gamma}$.

Proof. By using Lemma 1, we see that both sides of each proposed identity transform in the same way under G . Each identity asserts the equality of polynomials with integral coefficients on V and hence it suffices to verify that the identity holds at w . This may be done by direct calculation. \square

5. Covariant associative algebras

Let k be a field of characteristic zero. Denote by $\mathcal{E}_n(k)$ the set of isomorphism classes of commutative, associative k -algebras of dimension n over k and let $\mathcal{E}_n^{\text{sep}}(k) \subset \mathcal{E}_n(k)$ denote the subset of separable k -algebras. In [12], a one-to-one correspondence

$$G_k \setminus V_k^{\text{ss}} \leftrightarrow \mathcal{E}_5^{\text{sep}}(k)$$

is described in terms of Galois cohomology. In [5,6] a map

$$G_k \setminus V_k \rightarrow \mathcal{E}_5(k),$$

which extends this correspondence is described explicitly in terms of covariants of (G, V) . We shall briefly recall the definition of this map below. After that we give a

similar construction of a map

$$G_k \setminus V_k \rightarrow \mathcal{E}_6(k)$$

and identify the sextic algebra corresponding to a point in V_k^{ss} in terms of the quintic algebra corresponding to the same point.

Let k be a commutative ring with 1. If $v \in V_k$ then let $\tilde{R}_v = k \oplus \text{Aff}(4)_k^*$ and introduce a k -bilinear multiplication $\{\cdot\}_v$ on \tilde{R}_v by specifying that $(1, 0)$ is the multiplicative identity and

$$\{(0, f_i^*) \cdot (0, f_j^*)\}_v = \left(D_{ij}(v), \sum_k C_{ij}^k(v) f_k^* \right)$$

for $i, j = 1, \dots, 4$. It is shown in Section 5 of [5] that \tilde{R}_v with this multiplication and its usual addition is a commutative, associative k -algebra whose isomorphism class depends only on the G_k -orbit of v . Since \tilde{R}_v is free as a k -module, it has a well-defined discriminant, which is (the class of) $5^8 P(v)$. In particular, if k is a field of characteristic zero and $v \in V_k^{\text{ss}}$ then \tilde{R}_v is a separable quintic k -algebra. It is shown in Section 2 of [6] that the map $[v] \mapsto [\tilde{R}_v]$ coincides with the map constructed via Galois cohomology in [12].

If k is a commutative ring with 1 and $v \in V_k$ then let $\tilde{S}_v = k \oplus \text{Aff}(5)_k$ and introduce a k -bilinear multiplication $\{\cdot\}_v$ on \tilde{S}_v by specifying that $(1, 0)$ is the multiplicative identity and

$$\{(0, e_\alpha) \cdot (0, e_\beta)\}_v = \left(\Delta^{\alpha\beta}(v), \sum_\gamma \Gamma_\gamma^{\alpha\beta}(v) e_\gamma \right).$$

Proposition 2. *Let k be a commutative ring with 1 and $v \in V_k$. With the usual addition and the multiplication $\{\cdot\}_v$, \tilde{S}_v is a commutative, associative k -algebra whose isomorphism class depends only on the G_k -orbit of v . As a k -module, \tilde{S}_v is free of rank six and the discriminant of \tilde{S}_v is (the class of) $2^{12} 3^{10} P(v)^3$.*

Proof. The commutative and associative laws on \tilde{S}_v amount to various tensorial polynomial identities in $\Gamma_\gamma^{\alpha\beta}(x)$ and $\Delta^{\alpha\beta}(x)$ in the ring $\mathbb{Z}[V]$. If $\bar{\mathbb{Q}}$ denotes the field of algebraic numbers then it suffices to verify that these identities hold at all points of $V_{\bar{\mathbb{Q}}}$. Since the identities are tensorial, this will follow if the identities hold at $w \in V_{\mathbb{Z}}$. Thus it suffices to show that \tilde{S}_w is a commutative, associative ring. On calculation, we find that the values of $\Gamma_v^{\mu\lambda}(w)$ are

$$\Gamma_1(w) = \begin{pmatrix} -2 & 2 & 2 & 4 & -2 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2(w) = \begin{pmatrix} 0 & 2 & 0 & 3 & -3 \\ 2 & -2 & -1 & -2 & 1 \\ 0 & -1 & 0 & 0 & 3 \\ 3 & -2 & 0 & 0 & 0 \\ -3 & 1 & 3 & 0 & 0 \end{pmatrix},$$

$$\Gamma_3(w) = \begin{pmatrix} 0 & 0 & 2 & 3 & -3 \\ 0 & 0 & -1 & 0 & 3 \\ 2 & -1 & -2 & -2 & 1 \\ 3 & 0 & -2 & 0 & 0 \\ -3 & 3 & 1 & 0 & 0 \end{pmatrix}, \quad \Gamma_4(w) = \begin{pmatrix} 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 3 & 2 & -3 \\ 0 & 3 & 0 & 2 & -3 \\ -1 & 2 & 2 & 2 & -2 \\ 3 & -3 & -3 & -2 & 0 \end{pmatrix},$$

$$\Gamma_5(w) = \begin{pmatrix} 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 & -2 \\ -1 & -1 & -1 & -2 & 2 \end{pmatrix}$$

and those of $\Delta^{\alpha\beta}(w)$ are

$$\Delta(w) = \begin{pmatrix} 8 & -4 & -4 & -2 & -2 \\ -4 & 8 & 2 & 4 & -2 \\ -4 & 2 & 8 & 4 & -2 \\ -2 & 4 & 4 & 8 & -4 \\ -2 & -2 & -2 & -4 & 8 \end{pmatrix}.$$

Let L be the \mathbb{Z} -submodule of the ring \mathbb{Z}^6 spanned by the vectors

$$1 = (1, 1, 1, 1, 1, 1),$$

$$v_1 = (2, 2, 2, 2, -4, -4),$$

$$v_2 = (2, 2, -4, -4, 2, 2),$$

$$v_3 = (2, -4, 2, -4, 2, 2),$$

$$v_4 = (4, -2, -2, -2, -2, 4),$$

$$v_5 = (-2, -2, -2, 4, 4, -2).$$

Then L has rank 6 and one may verify that L is a subring of \mathbb{Z}^6 and that the structure constants of L with respect to the ordered basis $1, v_1, \dots, v_5$ agree with $\Gamma(w)$ and $\Delta(w)$. For example,

$$v_1 v_2 = -4 \cdot 1 + 2v_1 + 2v_2,$$

which agrees with the values $\Delta^{12}(w) = -4$, $\Gamma_1^{12}(w) = \Gamma_2^{12}(w) = 2$ and $\Gamma_\gamma^{12}(w) = 0$ for $\gamma = 3, 4, 5$.

The tensorial polynomial identity $\sum \Gamma_{\alpha}^{\alpha\beta}(x)=0$ holds at w , by inspection, and hence holds in $\mathbb{Z}[V]$. It follows that, in the k -algebra \tilde{S}_v , we have $\text{tr}(e_{\alpha})=0$ for all α . Thus the matrix of the trace form of \tilde{S}_v with respect to the ordered basis $1, e_1, \dots, e_5$ is

$$\begin{pmatrix} 6 & 0 \\ 0 & 6\Delta(v) \end{pmatrix}$$

and the determinant of this is $6^6 \det(\Delta(v)) = 2^{12} 3^{10} P(v)^3$, by part (3) of Lemma 3.

Finally, we must show that the isomorphism class of \tilde{S}_v depends only on the G_k -orbit of v . Suppose that $(h, g) \in G_k$ and set $v' = (h, g)v$. Define a k -linear map $\Phi_{(h,g)}: \tilde{S}_v \rightarrow \tilde{S}_{v'}$ by $\Phi_{(h,g)}((1, 0)) = (1, 0)$ and

$$\Phi_{(h,g)}((0, e_{\alpha})) = \left(0, \det(h)^{-3} \det(g)^{-5} \sum_{\gamma} g_{\gamma}^{\alpha} e_{\gamma} \right). \quad (4)$$

A routine calculation, making use of Lemma 1, reveals that $\Phi_{(h,g)}$ is a k -algebra homomorphism and $\Phi_{(h,g)}^{-1} = \Phi_{(h^{-1}, g^{-1})}$. Thus $\tilde{S}_v \cong \tilde{S}_{v'}$ as k -algebras. \square

Lemma 4. *Let k be a field of characteristic 0 and regard w as an element of V_k . Then $\tilde{S}_w \cong k^6$ and if we set*

$$12\pi_1 = (2 + 2e_1 + e_2 + e_3 + e_4 + e_5),$$

$$12\pi_2 = (2 + e_2 - e_3 - e_4 - e_5),$$

$$12\pi_3 = (2 - e_2 + e_3 - e_4 - e_5),$$

$$12\pi_4 = (2 - e_2 - e_3 + e_4 + e_5),$$

$$12\pi_5 = (2 + e_2 + e_3 - e_4 + e_5),$$

$$12\pi_6 = (2 - 2e_1 - e_2 - e_3 + e_4 - e_5),$$

then $\{\pi_1, \dots, \pi_6\}$ is the complete set of primitive orthogonal idempotents in \tilde{S}_w .

Proof. It can be verified by direct calculation, using the values of $\Gamma_v^{\mu\lambda}(w)$ and $\Delta^{\mu\lambda}(w)$ given in the course of the proof of Proposition 2, that π_1, \dots, π_6 are pairwise orthogonal idempotents. Since $\dim_k(\tilde{S}_w) = 6$, it follows that these idempotents are also primitive and that there are no other primitive idempotents in \tilde{S}_w . The Peirce decomposition associated with the π_j implies that $\tilde{S}_w \cong k^6$. \square

The proof of the main theorem requires some results from non-abelian Galois cohomology. This theory will also be used in Section 6. Since we must fix our conventions somehow, we shall adopt those used in [10]; see particularly Section 5 of Chapter I

and Section 1 of Chapter III of this work. To prepare for the proof, we now recall various facts.

Let k be a field of characteristic zero and \bar{k} an algebraic closure of k , henceforth fixed. Let $A_0 = k^n$, as a k -algebra, and suppose that A is a separable commutative k -algebra of dimension n over k . Then $\bar{k} \otimes_k A \cong \bar{k} \otimes_k A_0$ and so A is a k -form of A_0 . It is well-known that the set of such forms is in one-to-one correspondence with the Galois cohomology set $H^1(k, \text{Aut}(A_0))$. Now, any k -algebra automorphism of A_0 must permute the set of primitive idempotents in A_0 and from this we obtain an isomorphism $\text{Aut}(A_0) \cong \mathfrak{S}_n$, where \mathfrak{S}_n is the symmetric group on n symbols, well-defined up to conjugacy. Moreover, since all these automorphisms of A_0 are rational over k , $\text{Gal}(\bar{k}/k)$ acts trivially on $\text{Aut}(A_0)$. Thus, $H^1(k, \text{Aut}(A_0))$ may be identified with the set of conjugacy classes of homomorphisms from $\text{Gal}(\bar{k}/k)$ to \mathfrak{S}_n . Concretely, given A , we find n primitive orthogonal idempotents π_1, \dots, π_n in $\bar{k} \otimes_k A$. An element $\sigma \in \text{Gal}(\bar{k}/k)$ acts on $\bar{k} \otimes_k A$ by $\sigma(r \otimes a) = \sigma(r) \otimes a$ and permutes the set π_1, \dots, π_n . Mapping σ to this permutation gives a homomorphism $\text{Gal}(\bar{k}/k) \rightarrow \mathfrak{S}_n$, well-defined up to conjugacy, and this class of homomorphisms corresponds to the algebra A .

In this framework, it is easy to describe the sextic resolvent algebra of a separable quintic algebra. It is well-known that there is a single conjugacy class of monomorphisms $\Sigma: \mathfrak{S}_5 \rightarrow \mathfrak{S}_6$ such that the image of Σ acts transitively on the set $\{1, \dots, 6\}$. These monomorphisms arise, for example, from the permutation representation of \mathfrak{S}_5 on the six cosets of the normalizer of any Sylow 5-subgroup. If A is a quintic k -algebra and $f: \text{Gal}(\bar{k}/k) \rightarrow \mathfrak{S}_5$ is a representative of the class of homomorphisms corresponding to A then the class of the homomorphism $\Sigma \circ f: \text{Gal}(\bar{k}/k) \rightarrow \mathfrak{S}_6$ corresponds to a sextic k -algebra, the *sextic resolvent* of A . If A is, in fact, a quintic field then this is the sextic resolvent in the usual sense of Galois theory.

Next, we must describe the classification of $G_k \setminus V_k^{\text{ss}}$. For any $v \in V$, let G_v denote the isotropy group of v and G_v° its identity component. It is shown in [12] that there is a split short exact sequence

$$\{1\} \longrightarrow G_w^\circ \longrightarrow G_w \longrightarrow \mathfrak{S}_5 \longrightarrow \{1\} \quad (5)$$

of algebraic groups over k . For $g \in G_w$, let $[g]$ denote its image in \mathfrak{S}_5 . Given $v \in V_k^{\text{ss}}$, we may choose $g_v \in G_{\bar{k}}$ such that $g_v w = v$. For any $\sigma \in \text{Gal}(\bar{k}/k)$, $g_v^{-1} \sigma g_v \in G_{w\bar{k}}$ and the map $\sigma \mapsto [g_v^{-1} \sigma g_v]$ defines an element of $H^1(k, \mathfrak{S}_5)$. It is clear from the construction that this element depends only on the G_k -orbit of v and hence we obtain a map from $G_k \setminus V_k^{\text{ss}}$ to $H^1(k, \mathfrak{S}_5)$. It is proved in [12] that this map is a bijection. Since $\text{Gal}(\bar{k}/k)$ acts trivially on \mathfrak{S}_5 , the Galois cohomology set may once again be identified with the set of conjugacy classes of homomorphisms from $\text{Gal}(\bar{k}/k)$ to \mathfrak{S}_5 . In light of the construction made above, it follows that the set $G_k \setminus V_k^{\text{ss}}$ is in one-to-one correspondence with the set of isomorphism classes of separable quintic k -algebras. It is proved in [6] that this correspondence may be realized by mapping the orbit of v to the isomorphism class of the algebra \tilde{R}_v , whose construction was recalled above.

Theorem 1. *Let k be a field of characteristic zero and $v \in V_k^{\text{ss}}$. Then \tilde{S}_v is isomorphic to the sextic resolvent algebra of \tilde{R}_v .*

Proof. We first observe that, for any $g \in G_{\bar{k}}$, there is an isomorphism of \bar{k} -algebras

$$\Phi_g : \bar{k} \otimes_k \tilde{S}_v \rightarrow \bar{k} \otimes_k \tilde{S}_{gv}$$

defined as in (4). Let us take $v \in V_k^{\text{ss}}$ and choose $g \in G_{\bar{k}}$ such that $gw = v$. Then we have an isomorphism $\Phi_g : \bar{k} \otimes_k \tilde{S}_w \rightarrow \bar{k} \otimes_k \tilde{S}_v$ and $\Phi_g(\pi_j)$, $j = 1, \dots, 6$, are the primitive idempotents in $\bar{k} \otimes_k \tilde{S}_v$. If $\sigma \in \text{Gal}(\bar{k}/k)$ then

$$\begin{aligned} \sigma(\Phi_g(\pi_j)) &= \Phi_{\sigma g}(\pi_j) \\ &= \Phi_g \circ \Phi_{g^{-1}\sigma g}(\pi_j) \end{aligned}$$

and $g^{-1}\sigma g \in G_w$. Now G_w° necessarily acts trivially on the idempotents π_1, \dots, π_6 and, consequently, we may write this equation as

$$\sigma(\Phi_g(\pi_j)) = \Phi_g \circ \Phi_{[g^{-1}\sigma g]}(\pi_j).$$

It follows from this equation that the homomorphism $\text{Gal}(\bar{k}/k) \rightarrow \mathfrak{S}_6$ corresponding to the algebra \tilde{S}_v factors through the homomorphism $\text{Gal}(\bar{k}/k) \rightarrow \mathfrak{S}_5$ corresponding to the algebra \tilde{R}_v . All that remains is to check that the homomorphism $\mathfrak{S}_5 \rightarrow \mathfrak{S}_6$ given by

$$v \mapsto (\pi_j \mapsto \Phi_v(\pi_j)) \tag{6}$$

is in the conjugacy class of the homomorphism Σ described above.

Define $\tau, v \in G_{wk}$ by

$$\tau = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \right)$$

and

$$v = \left(\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

It follows from the results of [12] (see also [3, Lemma 6]) that the assignment $(12345) \mapsto \tau$, $(12) \mapsto v$ extends to a splitting of sequence (5). On calculation, the action of Φ_τ on the idempotents is found to be

$$\Phi_\tau(\pi_1) = \pi_3, \quad \Phi_\tau(\pi_3) = \pi_5, \quad \Phi_\tau(\pi_5) = \pi_6, \quad \Phi_\tau(\pi_6) = \pi_4, \quad \Phi_\tau(\pi_4) = \pi_1$$

and $\Phi_\tau(\pi_2) = \pi_2$; that of Φ_v is

$$\Phi_v(\pi_1) = \pi_3, \quad \Phi_v(\pi_2) = \pi_5, \quad \Phi_v(\pi_4) = \pi_6.$$

That is, under homomorphism (6), $(12345) \in \mathfrak{S}_5$ maps to $(13564) \in \mathfrak{S}_6$ and $(12) \in \mathfrak{S}_5$ maps to $(13)(25)(46) \in \mathfrak{S}_6$. In particular, the image of the homomorphism acts transitively on the set $\{1, \dots, 6\}$ and so the homomorphism lies in the conjugacy class of Σ . \square

6. The subgeneric orbit

Recall that Y denotes the hypersurface $P=0$ in V . Let Y' be the complement of the singular locus in Y . Then Y' is a G -invariant quasi-affine variety defined over \mathbb{Q} which we shall refer to as the *subgeneric orbit*; the terminology is justified by Proposition 3. Our main aim in this section is to discuss the classification of the G_k -orbits in Y'_k when k is a field of characteristic zero. Among other things, this will provide an interesting application of the construction of the algebra \tilde{S}_v when v is singular. Once we know that Y'_k is a single G_k -orbit, the problem of classifying $G_k \setminus Y'_k$ could also be approached using the inductive description of the orbits of a reductive group acting on a rational representation (see, for example, Section 3.2 of [13]). However, the description of the orbit set given in Theorem 2 provides perhaps the most direct method for solving the equivalence problem in specific cases.

Our first task is to show that Y' is a single orbit over an algebraically closed field of characteristic zero. It would be possible to approach this question in a computationally intensive manner using Ozeki's classification of the orbits in $V_{\mathbb{C}}$ [7] and the Lefschetz principle. Instead we give a conceptual proof that relies on the classification of the covariants of (G, V) given in [3]. This method could be adapted to other prehomogeneous vector spaces in which the generic point has finite isotropy group in $\mathrm{GL}(V)$.

Proposition 3. *Let K be an algebraically closed field of characteristic zero. Then Y'_K is a single G_K -orbit, which may be characterized as the unique Zariski dense G_K -orbit in Y_K .*

Proof. We first observe that G does not act effectively on V . The kernel of the action is the torus $A = \{(t^{-2}I_4, tI_5)\}$. Let \mathfrak{g} be the Lie algebra of G as a \mathbb{Q} -group and $\mathfrak{a} \subset \mathfrak{g}$ be the Lie subalgebra corresponding to A . For $v \in V$, we define a linear map $M(v): \mathfrak{g}/\mathfrak{a} \rightarrow V$ by $M(v)(Z) = Z \cdot v$, where the dot denotes the derived action of \mathfrak{g} on V .

For $g \in G$, let $\mathrm{Ad}(g): \mathfrak{g} \rightarrow \mathfrak{g}$ denote the adjoint map, that is, the differential of the map $G \rightarrow G$ given by $h \mapsto ghg^{-1}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the Lie algebra of $\mathrm{SL}(4) \times \mathrm{SL}(5)$ and note that there is an $\mathrm{Ad}(G)$ -invariant direct sum decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathbb{Q}Z_0 \oplus \mathfrak{g}_0,$$

where $Z_0 = (I_4, 0)$. Let us choose an ordered \mathbb{Q} -basis \mathcal{B} for $\mathfrak{g}/\mathfrak{a} \cong \mathbb{Q}Z_0 \oplus \mathfrak{g}_0$ whose first element is the class of Z_0 and whose remaining elements are the classes of a basis

for \mathfrak{g}_0 . Let $c(g)$ be the matrix of the induced map $\text{Ad}(g): \mathfrak{g}/\mathfrak{a} \rightarrow \mathfrak{g}/\mathfrak{a}$ with respect to \mathcal{B} and observe that $c(g)$ has the form

$$c(g) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \star & \\ 0 & & & \end{pmatrix}.$$

Let \mathcal{B}' be the basis $\{f_i \otimes (e_\alpha \wedge e_\beta) \mid \alpha < \beta\}$ of V ordered lexicographically with respect to (i, α, β) and let $b(g)$ be the matrix of the map $v \mapsto gv$ with respect to \mathcal{B}' . Let $m(v)$ be the matrix of the linear transformation $M(v)$ with respect to the bases \mathcal{B} and \mathcal{B}' . It is easy to see that $M(gv)(Z) = gM(v)(\text{Ad}(g^{-1})Z)$ and hence that $m(gv) = b(g)m(v)c(g)^{-1}$. The kernel of $M(v)$ is $\mathfrak{g}_v/\mathfrak{a}$, where \mathfrak{g}_v is the Lie algebra of the isotropy group of v . Since the identity component of the isotropy group of w is A , $M(w)$ has trivial kernel and hence $m(w)$ is an invertible matrix.

Let $\tilde{m}(v)$ be the classical adjoint of the matrix $m(v)$. The transformation law proved in the previous paragraph implies that we have $\tilde{m}(gv) = \chi(g)c(g)\tilde{m}(v)b(g^{-1})$, where $\chi(g) = \det(b(g))\det(c(g))^{-1}$ is a character of G . The rows of $\tilde{m}(v)$ are indexed by the basis \mathcal{B} . Let $\tilde{m}_1(v)$ be the first row (that is, the Z_0 row) of $\tilde{m}(v)$. From the expression for $\tilde{m}(gv)$ in terms of $\tilde{m}(v)$ and the form of $c(g)$ noted above, we see that $\tilde{m}_1(gv) = \chi(g)\tilde{m}_1(v)b(g^{-1})$. That is, the map $v \mapsto \tilde{m}_1(v)$ may be regarded as a relatively equivariant polynomial map from V to V^* , the dual of V , defined over \mathbb{Q} . The entries of $m(v)$ are linear polynomials in the coordinates of v and $m(v)$ is a 40×40 matrix. Thus the map $v \mapsto \tilde{m}(v)$ has degree 39. Moreover, $\tilde{m}(w)$ is invertible and so $\tilde{m}_1(w) \neq 0$.

The gradient of the relatively invariant polynomial P may also be regarded as a relatively equivariant map from V to V^* and, as such, it also has degree 39 and is defined over \mathbb{Q} . Since P is non-constant, the map $\nabla P: V \rightarrow V^*$ is non-zero. By Theorem 6 of [3], the space of relatively equivariant polynomial maps from V to V^* of degree 39 is one-dimensional, even over \mathbb{C} . Thus, there is a constant $C \in \mathbb{Q}$ such that $\tilde{m}_1(v) = C\nabla P(v)$. Since $\tilde{m}_1(w) \neq 0$, C is non-zero.

Now let

$$\mathcal{O} = \{v \in Y_K \mid \tilde{m}(v) \neq 0\},$$

a Zariski open subset of Y_K . Suppose that $z \in Y'_K$. Then, by Zariski's criterion, $\nabla P(z) \neq 0$ and, consequently, $\tilde{m}_1(z) \neq 0$. This shows that $Y'_K \subset \mathcal{O}$ and, in particular, that \mathcal{O} is non-empty. Now suppose that $z \in \mathcal{O}$. From the transformation law for $m(v)$ given above, it follows that $\det(m(v))$ is a relatively invariant polynomial of degree 40 and hence that $\det(m(v)) = C'P(v)$ for some $C' \in \mathbb{Q}$. Since $P(z) = 0$, the matrix $m(z)$ is singular, but, since $\tilde{m}(z) \neq 0$, $m(z)$ has a non-singular 39×39 minor. Thus the rank of $m(z)$ is 39. The column space of $m(z)$ is isomorphic to the tangent space of the G_K -orbit of z at z and hence the orbit is 39 dimensional. Since the orbit is contained in Y_K , it follows that the orbit is open and dense in Y_K . Because $z \in \mathcal{O}$ was arbitrary, we conclude that

\mathcal{O} is the G_K -orbit of z and that it is dense in Y_K . The set Y'_K is G_K -invariant and thus $Y'_K = \mathcal{O}$. \square

Now let k be a field of characteristic zero. If A is a finite-dimensional commutative k -algebra then we denote by $\mathfrak{N}(A)$ the nilradical of A . It is well known that

$$\mathfrak{N}(A) = \{a \in A \mid \text{tr}(ab) = 0 \text{ for all } b \in A\},$$

that $\mathfrak{N}(A)$ is an ideal of A and that $A/\mathfrak{N}(A)$ is a separable k -algebra. For $z \in Y'_K$, let $T_z = \tilde{S}_z/\mathfrak{N}(\tilde{S}_z)$. It follows from Proposition 2 and the remarks we have just made that T_z is a separable commutative k -algebra whose isomorphism class depends only on the G_K -orbit of z .

Henceforth, y will denote the point $y = y_1 + y_2 + y_3 + y_4$, where

$$y_1 = f_1 \otimes (e_3 \wedge e_4 - e_3 \wedge e_5 + e_4 \wedge e_5),$$

$$y_2 = -f_2 \otimes (e_2 \wedge e_3),$$

$$y_3 = f_3 \otimes (e_1 \wedge e_4 + 2e_2 \wedge e_4),$$

$$y_4 = -f_4 \otimes (e_1 \wedge e_5 + e_2 \wedge e_5).$$

We wish to explain the provenance of this point. In [7], Ozeki found the point $y' = y'_1 + y'_2 + y'_3 + y'_4$ in the subgeneric orbit over \mathbb{C} , where

$$y'_1 = f_1 \otimes (e_4 \wedge e_5),$$

$$y'_2 = f_2 \otimes (e_1 \wedge e_3 + e_2 \wedge e_5),$$

$$y'_3 = f_3 \otimes (e_1 \wedge e_4 + e_2 \wedge e_3),$$

$$y'_4 = f_4 \otimes (e_1 \wedge e_5 + e_2 \wedge e_4).$$

This point has a pleasing shape and was used in [3] as the standard representative of the subgeneric orbit. However, it is unsuitable for use in the current context because $\text{Gal}(\bar{k}/k)$ does not always act trivially on the component group $G_{y'}/G_{y'}^\circ$. The points y and y' are related by $y = py'$, where

$$p = \left(\begin{pmatrix} \frac{-1}{(2\omega+1)} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{\omega^2}{3} & \frac{\omega}{3} & \frac{1}{3} \\ 0 & \frac{\omega}{3} & \frac{\omega^2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{(2\omega+1)}{3} & \frac{-(2\omega+1)}{3} & 0 & 0 & 0 \\ \omega & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \omega & \omega^2 \\ 0 & 0 & 1 & \omega^2 & \omega \end{pmatrix} \right)$$

with ω a primitive cube root of unity.

Lemma 5. For all $z \in Y'_K$ we have $\dim_k(T_z) = 3$. The algebra T_y is isomorphic to k^3 and if we set

$$6\pi_1 = 2 - e_2,$$

$$6\pi_2 = 2 - e_1 - e_2,$$

$$6\pi_3 = 2 + e_1 + 2e_2,$$

then $\{\pi_1, \pi_2, \pi_3\}$ is the set of primitive idempotents in T_y .

Proof. Since $\bar{k} \otimes_k T_z \cong \bar{k} \otimes_k T_y$ for all $z \in Y'_k$, the first claim follows from the second. On calculation, the matrix of the trace form of \tilde{S}_y with respect to the standard basis is found to be the block sum

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 144 & -72 \\ 0 & -72 & 48 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and, consequently, $\mathfrak{N}(\tilde{S}_y)$ is the k -span of $\{e_3, e_4, e_5\}$ and $\dim_k(T_y) = 3$. By abuse of notation, we write $1, e_1, e_2$ for the classes of these elements in T_y . Further calculation shows that the multiplication in T_y satisfies the identities

$$e_1^2 = 24 - 6e_1 - 12e_2,$$

$$e_1e_2 = -12 + 2e_1 + 6e_2,$$

$$e_2^2 = 8 - 2e_2.$$

From this it is easy to verify that π_1, π_2 and π_3 are mutually orthogonal idempotents in T_y and, since $\dim_k(T_y) = 3$, they are necessarily a complete set of primitive idempotents. \square

Lemma 6. *There is a split short exact sequence of k -groups*

$$\{1\} \longrightarrow G_y^\circ \longrightarrow G_y \longrightarrow \mathfrak{S}_3 \longrightarrow \{1\} \quad (7)$$

and $\text{Gal}(\bar{k}/k)$ acts trivially on $G_y/G_y^\circ \cong \mathfrak{S}_3$. If $A = \{(t^{-2}I_4, tI_5)\}$ and

$$B = \{\text{diag}(t^{-2}, t^{-1}, t^{-1}, t^{-1}), \text{diag}(1, 1, t, t, t)\}$$

then the map $(a, b) \mapsto ab$ is an isomorphism from $A \times B$ to G_y° . The group elements

$$\tau = \left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right)$$

and

$$v = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right)$$

lie in G_y and the homomorphism sending $(23) \in \mathfrak{S}_3$ to τ and $(123) \in \mathfrak{S}_3$ to v splits sequence (7).

Proof. The isotropy group of y' was given in [3, Lemma 7], and since $y = py'$, the isotropy group of y may be determined from this. The relations $\tau^2 = v^3 = (\tau v)^2 = 1$ may be verified directly and they show that the subgroup of G_y generated by τ and v is isomorphic to \mathfrak{S}_3 . One particular isomorphism is given by $(23) \mapsto \tau$, $(123) \mapsto v$, and this establishes the final claim. \square

Theorem 2. *There is a one-to-one correspondence*

$$G_k \setminus Y'_k \leftrightarrow \mathcal{O}_3^{\text{sep}}(k).$$

The correspondence may be realized by mapping the orbit of $z \in Y'_k$ to the isomorphism class of T_z .

Proof. The first claim follows on combining arguments derived from [2,4]. We briefly sketch the reasoning. If $z \in Y'_k$ then there is some $g \in G_k$ such that $gy = z$. For any $\sigma \in \text{Gal}(\bar{k}/k)$, $g^{-1}\sigma g \in G_y$ and the map $\sigma \mapsto g^{-1}\sigma g$ defines an element of $H^1(k, G_y)$ whose image in $H^1(k, G)$ is the trivial class. It is easy to reverse the argument to show that there is a one-to-one correspondence between $G_k \setminus Y'_k$ and the kernel of the map

$$H^1(k, G_y) \rightarrow H^1(k, G).$$

Since $G = \text{GL}(4) \times \text{GL}(5)$, the Galois cohomology set $H^1(k, G)$ is trivial (see [10, Lemma 1 of Chapter 3]) and so $G_k \setminus Y'_k$ is in one-to-one correspondence with $H^1(k, G_y)$. The group G_y° is a product of tori and so $H^1(k, G_y^\circ)$ is also trivial. The long cohomology sequence associated to (7) then implies that the map

$$H^1(k, G_y) \rightarrow H^1(k, \mathfrak{S}_3) \tag{8}$$

is injective. Moreover, because (7) is split, the map (8) is surjective. Thus $G_k \setminus Y'_k$ is in one-to-one correspondence with $H^1(k, \mathfrak{S}_3)$. In light of our discussion of $H^1(k, \mathfrak{S}_n)$ in Section 5, the first claim follows.

In order to establish the second claim it suffices, as in the proof of Theorem 1, to show that the permutations of the set $\{\pi_1, \pi_2, \pi_3\} \subset T_y$ induced by the maps Φ_τ and Φ_v are (23) and (123), respectively. This is simply a matter of calculation. \square

Corollary 1. *Suppose that k is a field of characteristic zero that does not contain ω , a primitive cube root of unity. Then, under the correspondence of Theorem 2, Ozeki's point y' corresponds to the algebra $k \oplus k(\omega)$. In particular, $Y'_{\mathbb{R}}$ is the union of two $G_{\mathbb{R}}$ -orbits, one containing y and the other containing y' .*

Proof. It is easy to calculate $\Delta(y')$ and $\Gamma(y')$ directly. When we do so, we find that the matrix of the trace form of $\tilde{S}_{y'}$ with respect to the standard basis is the block sum

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 6^3 \\ 0 & 6^3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so $\mathfrak{N}(\tilde{S}_{y'})$ is the k -span of $\{e_3, e_4, e_5\}$. The classes of $1, e_1, e_2$ in $T_{y'}$ are found to satisfy the identities

$$e_1^2 = 6e_2, \quad e_1e_2 = 36, \quad e_2^2 = 6e_1.$$

If $18\pi_1 = 6 + e_1 + e_2$ and $18\varepsilon = 12 - e_1 - e_2$ then π_1 and ε are orthogonal idempotents in $T_{y'}$. Define $\theta \in T_{y'}$ by $18\theta = -6 - e_1 + 2e_2$. One verifies by calculation that $\varepsilon\theta = \theta$ and $\theta^2 + \theta + \varepsilon = 0$. From these identities it follows that $T_{y'} \cong k \oplus k(\omega)$ and this gives the first claim.

The separable cubic \mathbb{R} -algebras are \mathbb{R}^3 and $\mathbb{R} \oplus \mathbb{C}$ and hence $Y'_{\mathbb{R}}$ is the union of two $G_{\mathbb{R}}$ -orbits. We have $T_y \cong \mathbb{R}^3$ and $T_{y'} \cong \mathbb{R} \oplus \mathbb{R}(\omega) = \mathbb{R} \oplus \mathbb{C}$. Hence y represents one orbit and y' the other. \square

The reader may wonder about the structure of the separable k -algebra $U_z = \tilde{R}_z / \mathfrak{N}(\tilde{R}_z)$ derived from the quintic algebra \tilde{R}_z when $z \in Y'_k$. By using the methods explained in this section, it is easy to show that $U_z \cong k \oplus T_z$. Thus, T_z and U_z determine one another up to non-canonical isomorphism when $z \in Y'_k$.

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